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Recent Experimental Studies of the Mechanical

Response of Inelastic Solids to Rapidly

Changing Stresses

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## Abstract

The role of experiment in elucidating the dynamic
mechanical behavior of inelastic solids is discussed with
particular emphasis on the type of results which can be obtained
from wave propagation experiments. Recent experimental results
obtained by the author and his co-workers are reviewed as is work
by other investigators in the field. Viscoelastic waves, plastic
waves and shock waves are considered and in particular a recent
observation by the author of the generation of tensile shock waves
in stretched rubber is described. The fracture behavior
of brittle plastics under stress-wave loading, and some recent
work by Phillips on stress waves produced by tensile fractures are
mentioned.

## Introduction

Before reviewing experimental advances in the field of
stress wave propagation in anelastic solids it is perhaps
appropriate to discuss the reasons that such experiments are
carried out and what type of information they can be expected to
yield.  Elastic solids are assumed to behave in a completely
linear manner where the components of strain are postulated to be
linear combinations of the components of stress; it is found that
for sufficiently small deformations the experimental behavior of
many real solids closely corresponds with this assumption.  It
could be argued that even if this were not found to be so  in
nature, the theory of linear elasticity would still be worthwhile
as a mathematical study, in that the physical assumptions are so
simple that their mathematical consequences should be investigated
purely as a problem in mathematical analysis.  It might, of course,
be alternatively argued that applied mathematics should be
concerned with a description of the real world, and that the only
justification for work on elastic theory is that it describes
real situations; if mathematical problems are to be studied per se,
there are many more promising fields.

Fortunately, however, this question does not arise and the
theory of mathematical elasticity is a well established discipline
which has yielded many results which are essential to the design
of engineering structures.  Now the mathematical problems set by
elastic theory are always difficult and often intractable, and the
field of experimental stress analysis has arisen as a result of

this. Experimental work is used as a form of analogue computer
which produces numerical answers to problems for which exact
analytical solutions are not known. The experimental technique
of this type which has received perhaps the greatest attention
is photoelasticity — this depends on the observation that for
many materials the birefringence produced by elastic deformation
is linearly related to the applied shear stress. In using this
technique it is important to match the value of the Poisson's
ratio of the material used for the model to that of the structure
under investigation. It is also generally found to be limited to
two-dimensional stress-distributions, although in recent years
the technique has been extended by frozen-stress methods. With
the aid of these, some three-dimensional problems can be tackled;
the use of photoelastic coatings has also led to considerable
experimental advances. There are many books on the theory and
practice of this subject of which the classic by Coker and
Filon [1] and the treatises by Jessop and Harris [2], Frocht [3],
and Durelli and Riley [4] should be listed.

The use of electrical strain gages enables one to measure
the strain in a model of the same material as the structure to be
analyzed and the 'Handbook of Experimental Stress Analysis' [5]
gives excellent accounts of both this technique and that of photo-
elasticity.

When we come to dynamic problems which involve only
elastic deformations, most of the above considerations apply with
equal force, and although one more physical parameter, namely the

density, is now involved, theoretical solutions can be written
down for several simple problems.  Here again, the reason for
carrying out experiments is to obtain numerical solutions to
problems which are mathematically intractable, and the experi-
mental apparatus may again be regarded as an analogue computer.

Now when departures from exact elastic behavior take place,
the nature of the problems becomes quite different; theoretical
predictions are possible only if the inelastic behavior can be
specified mathematically and it is found that the way in which
materials deviate from perfect behavior varies from solid to
solid.  The deviations are found to be of two types.  First, as a
result of mechanical yielding (or because the magnitude of the
strains is becoming comparable with unity), the simple Hookean
relation between stress and strain is found no longer to be valid
and the strain becomes a non-linear function of the stress.
Furthermore the value of the strain may cease to be a univalued
function of the stress, and depends upon the particular path by
which the final stress was reached.  These are the types of
deviations which are dealt with in the mathematical theory of
plasticity and in the mathematical study of large rubber-like
deformations.  In these theories it is, however, assumed that, so
long as inertia effects can be neglected, the stress-deformation
relation does not depend on the rate of loading and the stress-
strain curve is the same for all straining rates.  These theories
give predictions which are found to approximate very closely to
the results observed experimentally, on the one hand with metals
and on the other with some rubbers.

Now in addition to this type of deviation many solids show time-dependent stress-strain behavior, thus if a constant load is maintained the strain is often found to increase steadily with time; this phenomenon is known as creep. Alternatively, if the material is deformed and the deformation is held at a fixed value, the stress is found slowly to decrease, this is a phenomenon known as stress relaxation. Such materials are termed viscoelastic and for small deformations many of these materials are found to obey Boltzmann's Principle of Linear Superposition.

This principle states that if a stress history $\sigma_1(t)$ produces a strain history $\epsilon_1(t)$ in a specimen, and a stress history $\sigma_2(t)$ produces a strain history $\epsilon_2(t)$ in the specimen then if the combined stress history $\sigma_1(t)+\sigma_2(t)$ is applied, the strain history will be given by $\epsilon_1(t)+\epsilon_2(t)$. A particular application of this principle is that if a stress history is of the same shape but has twice the value at each instant, the resulting strain-time curve will have the same shape but twice the amplitude. Materials which obey Boltzmann's Principle are termed linearly viscoelastic, and the relation between stress and strain in them can be expressed, either in the form of a convolution integral or as a differential equation which relates the stress and strain in terms of the derivatives of these two quantities with respect to time.

Many high polymers and rubber-like solids are found to be linearly viscoelastic/and the theory of wave propagation through for small strains, solids of this type has been treated theoretically for quite some

time, although often mathematical models which are analytically convenient but physically unrealistic have been employed. Hunter gives a comprehensive review of work in this field up to about 1960 [6] and the subject has also been reviewed by the present author [7-9].

The theory of wave propagation through materials with a yield point, the stress-strain behavior of which is essentially non-linear but is not rate-dependent was first considered by Donnell in 1930 [10]. It has subsequently been developed by G.I. Taylor [11], von Karman and Duwez [12], Rakhmatulin [13], and White and Griffis [14]. Accounts of this theory can be found in books by Kolsky [15] and Goldsmith [16] as well as in review articles by Broberg [17], Abramson, Plass and Ripperger [18], Cristescu [19] and by Craggs [20]. This theory has been found to predict fairly closely the propagation of large amplitude waves through metals although, as discussed later, observations which must be due to rate effects have become increasingly apparent in recent years.

The theory of waves of finite amplitude in solids has received considerably less attention, but one aspect of it namely, the setting up of shock waves in solids, has been studied intensively by a number of theoretical and experimental schools, and a review of this work by Duvall [21] summarizes the advances made in the field.

Most real materials are found to be to some extent both non-linear and rate-dependent in their stress-strain behavior, and

although the two extreme cases, namely, plastic wave propagation
and linear viscoelastic wave propagation, approximate very closely
to the behavior of metals and polymers respectively, even here
deviations are observed. For plastic waves, for example, although
the velocity of the plastic wavefront agrees with the simple
theory, the distribution of plastic deformation is found not to
be in agreement with the theoretical predictions. Furthermore the
rate-independent theory predicts that each element of strain
travels with a velocity given by $(S/\rho)^{1/2}$ where S is the tangent
modulus at that strain. Bell [22] tested this hypothesis experi-
mentally by pre-straining a copper, wire in tension quasi-statically
and then sending an additional tensile pulse along it. Since the
wire was already well into the plastic range due to the static
loading, the pulse would be expected to travel, according to the
rate-independent theory, at a velocity considerably below the
elastic wave speed. Bell found, however, that the superimposed
pulse traveled at the velocity $(E/\rho)^{1/2}$ where E was Young's
modulus for copper, the modulus which should be applicable only
for very small strains. This result was confirmed by Sternglass
and Stuart [23] for copper and led to similar work by Alter and
Curtis [24] on lead specimens. The results of these experiments
seem to be explicable only in terms of a rate-dependent theory
and theories of this sort have been developed by Sokolovsky [25]
and Malvern [26],[27]. These theories, although based on highly
simplified models show the general effect of a velocity of
propagation for rapidly changing stresses of $(E/\rho)^{1/2}$, while the

quasi-static stress-strain curve is that observed experimentally. Although the mathematical treatments of wave propagation in such non-linear viscoelastic solids is necessarily involved, more theoretical work of this type would certainly prove useful.

When the amplitude of the stress pulses becomes sufficiently large, the specimen fails completely, and brittle or ductile fractures take place.  Interest in these problems was revived in the Second World War, and work along these lines has subsequently continued.  Accounts of the results of investigations on metals are given in the book by Rinehart and Pearson [28] and also a general discussion will be found in a recent review article by Rader and the present author [29].

## Linear Viscoelastic Solids

The relation between stress and strain for most solids is found to depend, to some extent, on the duration over which the stress is maintained.  Thus if a constant stress is applied to a solid specimen, the strain is, in general, found to increase gradually with time, and this phenomenon is called creep. Alternatively if a constant deformation is maintained the stress is found to decrease monotonically with time and this phenomenon is called stress-relaxation.  The magnitudes of these phenomena can be relatively very small, as is found to be the case for metals, or relatively quite large as is found for many soft plastics.  The creep and stress relaxation may be linearly pro-portional to the stress, or they may be  complicated functions of the stress and time.

When the creep is linearly proportional to the stress, the mathematical treatment is considerably simplified and the material is called a <u>linear viscoelastic solid</u> [see Gross [30]]. Thus if we first consider uniaxial extension, we have that a constant stress $\sigma$ applied at time t=0, produces a uniaxial strain $\varepsilon$ such that

$$\varepsilon = \sigma \, \psi(t) \tag{1}$$

where $\psi(t)$ is called the creep function.

If we have a loading history which consists of a series of constant loads $\Delta\sigma_1, \Delta\sigma_2, \Delta\sigma_3, \ldots, \Delta\sigma_n$, which can be either positive or negative, commencing at times $t_1, t_2, t_3, \ldots, t_n$ etc. then as a result of the principle of linear superposition, the strain at any time t is given by

$$\varepsilon(t) = \sum_{p=1}^{p=n} \Delta\sigma_p \, \psi(t-t_n) \tag{2}$$

and in the limit for a continuous loading history we have

$$\varepsilon(t) = \int_{\tau=-\infty}^{\tau=t} \psi(t-\tau) \, \frac{d\sigma}{d\tau} \, d\tau \tag{3}$$

This is one mathematical expression of Boltzmann's superposition principle. An alternative representation is to consider the specimen to have undergone a particular deformation history $\varepsilon(t)$ and express the stress $\sigma(t)$ as a convolution integral involving the strain and the stress relaxation function $\overline{\psi}(t)$ thus

$$\sigma(t) = \int_{\tau=-\infty}^{\tau=t} \overline{\psi}(t-\tau) \, \frac{d\varepsilon}{d\tau} \, d\tau \tag{4}$$

These methods of describing the stress-strain relations of linearly viscoelastic solids are physically the most satisfactory, but without some knowledge of the nature of the creep function $\psi(t)$ or the stress relaxation function $\overline{\psi}(t)$ they are not particularly useful in solving specific problems, and an alternative approach has been to express the stress-strain relations as linear differential equations involving the stress, the strain and their derivatives with respect to time. Thus for a single stress component $\sigma$ and a single strain component $\varepsilon$, we may write

$$P\sigma = Q\varepsilon \tag{5}$$

where P and Q are linear differential operators thus

$$P = a_o + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} + \ldots$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6)$

$$Q = b_o + b_1 \frac{d}{dt} + b_2 \frac{d^2}{dt^2} + \ldots$$

This treatment can readily be extended to three dimensions by defining the stress and strain deviators $s_{ij}$ and $e_{ij}$ by the relations

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \, \delta_{ij} \tag{7}$$

and

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3} \varepsilon_{kk} \, \delta_{ij} \tag{8}$$

(where $\delta_{ij}$ is the Kronecker delta thus $\delta_{ij} = 1$ when i=j and $\delta_{ij} = 0$ when i≠j). We can then write the stress-strain relations in terms of four linear operators; thus

$$Ps_{ij} = Q\,e_{ij} \tag{9}$$

and

$$P'\sigma_{ii} = Q'e_{ii} \tag{10}$$

where P and Q here describe the shear behavior, P' and Q' the bulk behavior.

The use of linear operators such as those given by these equations allow viscoelastic problems to be treated in terms of simple linear differential equations, and so long as the response of the viscoelastic material can be reasonably approximated to by a few terms in the operational polynomials, it is possible to obtain solutions of /specific dynamic problems in closed form, or at least to obtain numerical solutions. The use of finite polynomials for P and Q is exactly equivalent to the assumption that the material is behaving like a mechanical model composed of perfectly elastic springs and viscous dashpots;(for the latter, the velocity is assumed proportional to the applied force). The three simplest models of this type are the Maxwell model for which $P = a_o + a_1 \frac{d}{dt}$ and $Q = b_1 \frac{d}{dt}$, and this model corresponds to a Hookean spring and a viscous dashpot in series, the Kelvin-Voigt model which is a spring with a dashpot across it and for which $P = a_o$ and $Q = b_o + b_1 \frac{d}{dt}$, and the standard linear model which corresponds to either a Maxwell model with a spring across it or a Voigt model with a spring in series with it, and for which $P = a_o + a_1 \frac{d}{dt}$ and $Q = b_o + b_1 \frac{d}{dt}$. A number of dynamic problems, including longitudinal wave propagation along a thin filament have been treated by Lee and his co-workers [31] and the results of this

work give a general qualitative picture of the phenomena to be expected. As is discussed below, the quantitative agreement is not so satisfactory except for a few rather special solids.

Another way of treating linear viscoelastic behavior is to consider how a specimen responds to a deformation that varies sinusoidally with time. Since the material is assumed to obey a linear differential equation, the stress will oscillate at the same angular frequency as the strain, although not necessarily in phase with it. Thus for a uniaxial deformation of angular frequency $\omega$, we have if the strain $\varepsilon$ is

$$\varepsilon = \varepsilon_0 \cos \omega t \qquad (11)$$

the uniaxial stress is given by

$$\sigma = \sigma_0 \cos(\omega t + \delta) \qquad (12)$$

The response is then defined by the ratio of the stress amplitude to the strain amplitude, i.e. $E^* = \sigma_0/\varepsilon_0$, and also by the angular phase lag $\delta$, both of these quantities are in general frequency dependent.

These two parameters are comparatively simple to measure, and such measurements have been carried out by many workers over extremely wide ranges of frequency. These include Lethersich [32] who measured the response of specimens of several common polymers in torsional oscillation. Figure 1 shows a plot of his results for $\log_{10}(\tan \delta)$ against $\log_{10} \omega$ and the results are compared with the response expected for Maxwell, Kelvin-Voigt and standard linear solids, (the positions of these latter curves are arbitrary). On

the same plot are shown the measured response of two <u>organic</u> <u>glasses</u>, hydroxy pentamethylflavan and glycerol sextol pthalate. These materials are of comparatively low molecular weights and the points are based on measurements made by Benbow [33]. It may be seen  that although the behavior of polymers is badly out on a numerical basis from that of the model solids, both organic glasses correspond very closely to Maxwell solids over several decades of frequency.  Only  at higher frequencies do they too begin to flatten out like the high polymers.

The figure shows that if a simple model is to be taken for the high polymers, which are after all by far the most important viscoelastic solids, the assumption that tan $\delta$ is independent of frequency is a far better one than any of the simple mechanical models.

The theory of linear viscoelasticity shows that tan $\delta$ and $E^*$ are not independent functions and that if for example $E^*$ is known at all frequencies, tan $\delta$ can be calculated for all frequencies.  Similarly, if tan $\delta$ is known at all frequencies, $dE^*/d\omega$ at all $\omega$ can be determined, and hence if the value of $E^*$ is known at any one frequency $\omega_0$, $E^*$ at all frequencies can be determined.  In the particular case where tan $\delta$ is independent of $\omega$ over a large frequency range, the approximate relation

$$E^*(\omega) \simeq E_0(1 + \frac{2 \tan \delta}{\pi} \ln \omega/\omega_0) \tag{13}$$

may be shown to hold, where $E_0$ is the value of $E^*$ at some standard frequency $\omega_0$.

We now come to the propagation of an infinite train of sinusoidal waves of angular frequency $\omega$ along a filament. If at the origin, the stress $\sigma$ is defined by

$$\sigma = \sigma_o \cos \omega t \qquad (14)$$

the stress at any point a distance x from the origin is

$$\sigma = \sigma_o \exp(-\alpha x)\cos \omega(t-x/c) \qquad (15)$$

where $\alpha$ is the attenuation coefficient which is given by

$$\alpha = \frac{\omega \tan \frac{\delta}{2}}{c}, \qquad (16)$$

and c is the phase velocity which is given by

$$c = (E^*/\rho)^{\frac{1}{2}} \sec \frac{\delta}{2} \qquad (17)$$

$E^*$ and $\tan \delta$ are here assumed to have the values appropriate to the angular frequency $\omega$ . Now since linear superposition applies an expression can be written down for the propagation of a disturbance of any shape. Thus, if the strain at the origin $\varepsilon(0)$ is given by the Fourier integral

$$\varepsilon(0) = \int_o^\infty A(\omega)\exp i\omega t \, d\omega, \qquad (18)$$

the strain $\varepsilon(x)$ a distance x along the filament is then given by the expression

$$\varepsilon(x) = \int_o^\infty A(\omega)\exp[-\alpha x+i\omega(t-x/c)]d\omega \qquad (19)$$

where α and c are both real functions of ω whereas A(ω) is, in general, a complex function of ω.

In order to calculate the change in shape of any mechanical pulse as it progresses along a viscoelastic rod all that is required is a knowledge of how α and c vary with frequency over a sufficiently wide frequency range, and direct measurements of these quantities can be made by observing the phase and amplitude of sinusoidal waves travelling along filaments [e.g. Hillier and Kolsky [34] and Hillier [35]]. Alternatively the value of c and α as functions of ω can be obtained from the values of E* and tan δ using relations (16) and (17). Once the values of c and α are known over a sufficiently wide range of frequencies the Fourier integral can be approximated to by a Fourier sum of cosines and sines.

If we are able to use the approximate relation (13), (which is appropriate for most polymers at temperatures remote from their transition temperature) we find that for the phase velocity

$$c(\omega) \approx c_0 \left(1 + \frac{\tan \delta}{\pi} \ln(\omega/\omega_0)\right). \tag{20}$$

The shape assumed by a sharp pulse after it has travelled some distance along a rod of any such polymer can be obtained in non-dimensional form [see Kolsky [36]] and the 'universal shape' of a pulse can be calculated once and for all. The difference caused by variations in initial shape rapidly disappear, since these are produced by the high frequency components which are greatly

attenuated. Figure 2 shows this universal shape calculated theoretically and compares it with the shape of the pulse observed from the sharp blow produced by the detonation of a small explosive charge at the end of a polymethylmethacrylate rod after it has travelled 6 meters. It may be seen that the agreement is extremely satisfactory.

In such experiments as those described above the visco-elastic modulus involved is a generalized form of Young's modulus, E, but for most three-dimensional problems in wave propagation two time-dependent moduli are involved which correspond to the two elastic constants in three-dimensional elasticity. Physically the two most reasonable moduli to consider are those which corre-spond to the shear modulus and the bulk modulus, but experi-mentally the latter is rather difficult to measure. Recently Lifshitz and the author [37] have studied the propagation of spherical pulses in large blocks of plastics using an experimental set-up illustrated diagram in figure 3, and have found that the viscoelastic bulk modulus as well as the shear modulus is frequency dependent. For the three plastic studied namely, polyethylene, polymethylmethacrylate and polystyrene, the value of tan $\delta$ for volume changes was found to be about one fifth of its value for shear deformations, but it is too early to say whether this is a general rule for polymers of this type, or whether the similarity of the results for the three polymers is to some extent coincidental. The observations that the bulk loss follows the shear loss over a large range of values does seem to give strong indication, however,

that what is observed as 'bulk viscosity' macroscopically is the
result of shear losses on a molecular or microscopic scale; for
a material composed of long chain molecules this is not
particularly surprising on physical grounds.  It should be
emphasized that for ordinary fluids the bulk losses are a tiny
fraction of the shear losses due to viscosity.

Figure 4 shows the observed pulse shape
in a block of polyethylene compared with the shapes predicted on
the basis of assumption that, (a) the bulk loss is zero, (b) the
bulk loss is one fifth of the shear loss and (c) that the bulk
loss is equal to the shear loss.  It may be seen that the agree-
ment observed between the experimental results and the curve
based on the second assumption is extremely close.

Very recently Tsai and the author [38] have observed the
pulses produced by the impact of steel balls as they propagated
over the surfaces of blocks of polyethylene and
polymethylmethacrylate.        The results of this work have
shown that the assumption that the ratio of the shear loss to the
bulk loss is a constant once again gives excellent predictions of
the shapes of the pulses propagated over the surfaces of the
specimens.

## Plastic Waves

The time dependence of the stress-strain relation for
most metals is very much less marked than it is for high polymers
and, as a first approximation, the assumption that the stress-strain

relation is time-independent is a perfectly satisfactory one for most purposes. The first treatment of the problem of wave propagation along a material with such characteristics was published by Donnell [10] in 1930. Donnell considered the propagation of a longitudinal step wave along a thin rod of a material which had a bilinear stress-strain curve, the first part of the curve was assumed to correspond to the elastic behavior, and the second to strain-hardening associated with plasticity. Donnell showed that under these conditions two wave fronts would propagate along the rod, the fastest one travelling at the elastic wave speed $(E/\rho)^{1/2}$ where E is Young's modulus for elastic deformations, and the second which has been termed a plastic front at a speed given by $(S/\rho)^{1/2}$ where S is the slope of the plastic portion of the curve.

The subject lay dormant for about ten years and only during the Second World War did a number of workers, all apparently unaware of Donnell's paper, reopen the question of plastic waves. In England, G.I. Taylor [11] tackled the problem, in this country it was von Karman and Duwez [12] and White and Griffis [14], while in the Soviet Union the problem was treated by Rakhmatulin [13]. Apparently, here again, there was little communication between these separate workers and only later were the results published. As mentioned in the introduction, the nature of plastic wave propagation has been reviewed by several authors [15-20] and the predictions of the rate-independent theory are found to be reasonably borne out in practice. A careful examination of the results of Duwez and

Clark [39] by Lee [31] has shown that the distribution of
permanent plastic strain does not quantitatively agree with that
observed experimentally, and although this was originally attributed
to the complicated reflections of the plastic wave during the
unloading process — a number of features and particularly the
absence of a sharp plastic wavefront led Lee to surmise that the
rate dependence of the stress-strain curve was responsible for the
experimentally observed deviations.  Malvern [26] developed a theory
to account for such deviations and has recently shown [27] that
the original calculations were subject to computational error and
were in any case taken for far  too late a time, the true picture being
**very**   much closer to the observed behavior.  Figure 5 shows the
curves obtained by Malvern for two times after the first applica-
tion of the load.  It may be seen that the plastic wavefront is
much less well-defined than in the rate-independent theory and the
plateaux of plastic strain do not persist so far down the wire.
This observation is in accord with the experimental observations
of many workers in the field including those of Duwez and Clark [39],
Douch and the author [40] observed similar effects with aluminum
and   copper specimens.  A year before Malvern's first paper
appeared, Sokolovsky [25] in the Soviet Union published a similar
type of visco-plastic theory and this led to the same equations as
those derived by Malvern.  He did not, however, carry out any
specific calculations of the effect of rate-dependence.

At about the same time Bell [22] tested the rate-independ-
ent theory in another way, he stretched a copper specimen and

while it was being drawn he propagated an additional incremental tensile pulse. According to the rate-independent theory such incremental pulses should travel at a velocity of $(S/\rho)^{1/2}$ where S is the tangent modulus, in fact Bell found they travelled at a velocity $c_o = (E/\rho)^{1/2}$, the elastic wave velocity, which, since the copper was well in its plastic range, was considerably higher than $(S/\rho)^{1/}$

As mentioned earlier these results were later confirmed for copper by Sternglass and Stuart [23] and for lead by Alter and Curtis [23]. In the work with lead, both the pre-load and the incremental load were dynamic. Since, however, this metal is notoriously rate-dependent and almost viscoelastic in its behavior, the reliability of the conclusions to be drawn from this work is to some extent doubtful. The theories of Malvern and Sokolovsky both predict that 'instantaneous' changes in stress will travel at the elastic wave speed and thus although the time-dependent non-linear behavior in both these theories is rather simplified, they do seem to explain the observed discrepancies of the shape of the plastic wavefront and the Bell effect very well.

In determining the stress-strain behavior of such non-linear time-dependent solids at these very high loading rates one is to some extent trapped in a vicious circle. Any attempts to make measurements in short times inevitably involves wave propagation, these cannot be interpreted unless the nature of the constitutive relation is known. Many attempts have been made to circumvent this difficulty, the earliest being by G.I. Taylor [41], by E. Volterra [42] and by the author [43].

These methods depend on the use of auxiliary steel bars which are themselves not strained beyond their elastic limits and along which elastic waves travel. Observations of these waves enables the stress-strain behavior of the inelastic specimen to be inferred. The method devised by the author (43) consists of inserting a cylindrical wafer of the material under investigation into a split 'Hopkinson Bar'. A mechanical pulse is then propagated along the bars and by observation of the elastic stress pulses in the two steel bars, the stress-time curve and the strain-time curve for the specimen can be obtained.

During the last twenty years this method has been used extensively for stress-strain measurements at very high rates of loading by workers including Hauser, Simmons and Dorn [44], Davies and Hunter [45], Lindholm [46], Malvern [47], and Ripperger [48]. As was first pointed out by the author [43], there are serious limitations to this experimental method. Thus, on the one hand if the specimen is long compared with the pulse length, wave propagation can occur along it and it is not subjected to constant longitudinal stress; if on the other hand the specimen is very short, frictional effects prevent uniform straining and barrelling may occur, although this can to some extent be reduced by suitable lubrication of the interfaces. Eubanks, Muster and Volterra [49] have considered the theory of the first effect, and Bell [50] has made a careful experimental study of some of the errors which might be expected to take place. Jahsmann [51] recently has investigated theoretically the likely expected errors

resulting from the wave propagation effects and he has shown that so long as the pulse length is several times the length of the specimen, the errors are of manageable proportions.

Some years ago Bell [52] devised an extremely elegant technique for making dynamic strain measurements. This method depended on a diffraction grating ruled on the specimen, he measured the angle of the diffracted beam and thus inferred the values of the strains. He has used this technique successfully in a number of dynamic stress-strain measurements and has shown that the rate effect for highly annealed and extremely pure metals is very small [53].

Most other workers using less highly annealed metals have observed rate effects although these are generally quite small in magnitude. Figure 6 shows a comparison of the 'static' and 'dynamic' stress-strain curves for pure annealed aluminum obtained by Douch and the author [40] and it may be seen that the rate effect is not large. Rate effects of the same order of magnitude were observed in these experiments with pure annealed copper, while a aluminum-silicon alloy showed no measurable rate effect. There have been numerous other measurements of rate effects in metals and they go back to the work of Clark and Datwyler [54] in 1938. The work of Manjoine and Nadai in 1940 [55] is still of some interest, and the excellent work of Campbell and Marsh [56] and Maiden and Green [57] should also be referred to. This by no means exhausts the list of workers in the field and reference should be made to a recent survey report by Lindholm and Bessey [58], to the Proceedings of the conference on the 'Mechanical Properties of Materials

at High Rates of Loading' edited by Lindholm [59], as well as to
the proceedings of the ASME symposium edited by Huffington [60].

The main conclusion which is derived from all these
investigations is that, except for steel, rate effects in metals
are relatively unimportant but can nevertheless sometimes result
in quite startling deviations such as the Bell effect. For most
purposes, however, it is sufficient to use a single 'dynamic'
stress-strain curve to account for the observed behavior, and even
if the 'static' curve is used the errors observed are not large.
Recently dynamic behavior under multi-axial loading has been
receiving much more attention, for example a paper on this subject
was presented by Lipkin and Clifton [61] at the last IUTAM Congress.

When we come to other non-linear time-dependent behavior
the experimental evidence is very much more sparse. Goldsmith and
Austin [62] have studied wave propagation in diorite rock specimens
and found that the loss for this material is of a frictional type
and that the modulus is not frequency sensitive. Calvit, Rader,
and Melville [63] performed some preliminary measurements on pulse
propagation along plasteline-clay rods and found that whereas
this material behaves like a linear viscoelastic solid for pulses
of small amplitude, at larger amplitudes the material flows
plastically like a metal. Bodner and the author [64] studied both
the free oscillations and the propagation of pulses in lead bars
and found that the logarithmic decrement was amplitude dependent
but the modulus was insensitive to frequency. This resulted in
the propagation of more or less symmetrical pulses, which decreased

in amplitude and broadened as they travelled along the lead bars.

Another type of non-linear behavior on which a large amount of experimental and theoretical effect has been expended, is the propagation of shock waves in solids. Most of this work is concerned with the propagation of large amplitude compression pulses from the detonation of explosive charges on blocks of the solids. As a result of the increasing value of the bulk modulus at these high pressures, shock fronts are producible, this work enables constitutive equations relating pressure and volume to be obtained, and as mentioned earlier, Duvall [21] gives an excellent summary of work of this type which has been carried out in recent years.

Another type of shock wave can occur in tension when the stress-strain curve is concave upwards, i.e. when the tangent modulus $d\sigma/d\varepsilon$ increases with increasing strain. Lee and Tupper [65] have discussed the theoretical basis for such shock wave generation in steel, where at large amplitudes the stress-strain curve becomes concave. In the book [15] by the author, it is pointed out that similar behavior might be expected in rubber, which also has a dynamic stress-strain curve which is concave upwards. Figure 7 shows the velocity of propagation of small sinusoidal waves along filaments of natural rubber as a function of the tensile pre-strain. These measurements were carried out by Hillier [66]. The large attenuation of high frequency waves prevents very sharp shock fronts from being set up in the material when it is in the unstretched state. If, however, the rubber is highly stretched to begin with and an additional tensile

pulse is propagated along it, a tensile shock wave may develop since the attenuation is then much lower. Mason [67] showed indications of such behavior in his studies of the free retraction of rubber specimens. Very recently the author has carried out some hitherto unreported experiments to investigate this effect. A specimen of natural rubber gum stock initially 1/2 inch square was stretched to five times its original length. At one end a portion 10 inches long was extended to 11 inches so that the strain in this small portion was 440%. This portion was held at this extra strain by a small piece of steel piano wire which was rapidly volatilised by the passing of a heavy electric current, a tensile pulse then travelled along the stretched specimen. The velocity-time profiles of the tensile pulse at a series of stations along the rubber was measured by affixing light wires to the rubber at various points along it and amplifying the currents induced in these as they cut the lines of force of strong uniform magnetic fields which were produced by permanent magnets. The technique was similar to that described by Efron and Malvern [68]. Figure 8 shows the velocity time profiles of the pulse as it travels along the rubber band. The shapes are given at 1, 3, 5, 7, and 9 ft. from the point of release. It can be seen that the front of the pulse becomes progressively steeper as it travels along the rubber specimen and is quite sharp at the 7 and 9 ft. stations. A paper is being prepared on this work and it is hoped that it will appear shortly.

## Stress Waves and Fracture

The earliest work on the fractures produced by stress waves was carried out by John Hopkinson in 1872 [69]. Further work was later carried out by his son Bertram Hopkinson [70] who was killed in the First World War. The subject was then neglected until comparatively recent times, but during and since World War II there has been a steady output of work on this subject. The book by Rinehart and Pearson [28] and a recent review article by Rader and the author [29] cover many of these investigations, which have all been concerned with the nature of the fractures produced in stress-wave loading. Recently the complementary problem namely the stress pulses produced by brittle fractures has been receiving attention at Brown University.

The first work on this was carried out by Tsai and the author [71] who studied the surface waves produced by the impacts of steel balls on glass plates. It was shown that when fracture occurs the outgoing stress pulse shows 'spikes' which were otherwise absent, and that the position of these spikes agrees with the theoretical analysis of the problem of a Hertzian impact.

More recently J. Phillips [72] has studied the problem of the stress pulses generated in glass rods when they are broken in tension and in flexure, a small notch was first made on the surface of the rod to initiate fracture. The analysis of the flexure problem is extremely complex, but for the case of fracture in tension, Phillips was able to show that two pulses are generated, one a longitudinal pulse, (this is to be expected since

the tensile force goes rapidly from its initial value to zero)
and also a flexural pulse; this results from the bending moment
set up by the asymmetrical nature of the stress release.
Figure 9 shows a comparison of the experimental curve observed by
Phillips with a strain gage mounted on the surface of a glass rod
at a distance of 6.2 inches from the position of the fracture
initiation.  The strain gage was mounted in line with the notch
so that both longitudinal and flexural pulses could be observed.
These experimental results are compared with the calculations
made on the basis of the propagation of elastic waves along the
rod.  He assumed that the fracture spread out from the point of
initiation at a constant velocity equal to 0.38 times the velocity
$c_o$ of longitudinal elastic waves along the rod.  $c_o = (E/\rho)^{1/2}$
and that the stress in the unbroken region remained unchanged.
Phillips further assumed that the longitudinal pulse travelled
along the rod without change in form, while the equation governing
the propagation of flexural waves was that developed by
Timoshenko [73], which R.M. Davies [74] has shown gives results
extremely close to the exact Pochhammer-Chree theory for
cylindrical bars when the first mode of propagation is considered.
In order to take into account all the boundary conditions at the
plane of fracture, Phillips found he had to use both branches of
the solutions of the Timoshenko equation.  It may be seen that
the agreement between experiment and theory is remarkably good.

## Conclusion

There are two ultimate reasons for carrying out experiments on the dynamic mechanical behavior of anelastic solids. The first is to obtain numerical data which can be used to predict the mechanical behavior of systems under conditions that have not hitherto been studied. In order to do this, mathematical models have to be set up which, while simple enough to make the mathematics tractable, incorporate enough of the features of physical reality to give reasonably numerical predictions of the quantities in regions outside the field of measurement, and which provide good interpolations or reasonable extrapolations of the physical facts. It is often very tempting to believe in the physical existence of these models as for example in the existence of real springs and viscous dashpots in linear viscoelasticity, but generally nature is nothing like so simple as this and a realistic physical model of even something very much less complicated, like liquid benzene, has never really been achieved.

The other reason for carrying out such measurements is in the hope of obtaining a better understanding of the molecular processes which are responsible for the observed macroscopic behavior. This approach has been extremely successful in elucidating the behavior of gases but has been markedly less so in treating the mechanical behavior of liquids and solids. The theory of dislocations first developed by G.I. Taylor [75] is a notable exception to this general rule but even here the quantitative prediction of dynamic mechanical properties is

still a long way off.

In conclusion it would perhaps be appropriate to call for more experimental work even if it is at the expense of the vast amount of theory which is so often based on oversimplified models of mechanical behavior, and to make a plea for some workers in the field to leave their desks and enter the harsh but exciting world of the laboratory.

## Acknowledgements

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## Legends for Figures

1. Measured values of tan δ compared with experimental results.
   A – polyethylene, B – polymethylmethacrylate, C – ebonite,
   D – polystyrene, E – hydroxypentamethylflavan, F – glycerol
   sextol phtalate, M – Maxwell solid, K – Kelvin solid,
   SL – standard linear solid.

2. Comparison  of measured universal pulse shape with
   experimentally observed pulse which has travelled through
   600 cm. of polymethylmethacrylate.

3. Experimental set-up for pulse propagation in blocks of
   plastics.

4. Comparison of observed pulse shape with that theoretically
   predicted for different values of bulk loss. A-zero bulk loss,
   B-bulk loss = 1/s(shear loss), C-bulk loss = shear loss.

5. Comparison of rate-dependent and rate-independent postulates
   in predicting plastic wave profiles (after Malvern).

6. Comparison of static and dynamic stress-strain curves for
   pure aluminum.

7. Dynamic modulus and tangent modulus of 'static' stress-strain
   curve of natural rubber as a function of longitudinal strain
   (after Hillier).

8. Stages in formation of tensile shock front in rubber specimen.

9. Strain gage observations on surface of glass rod resulting
   from tensile fracture. (after Phillips)
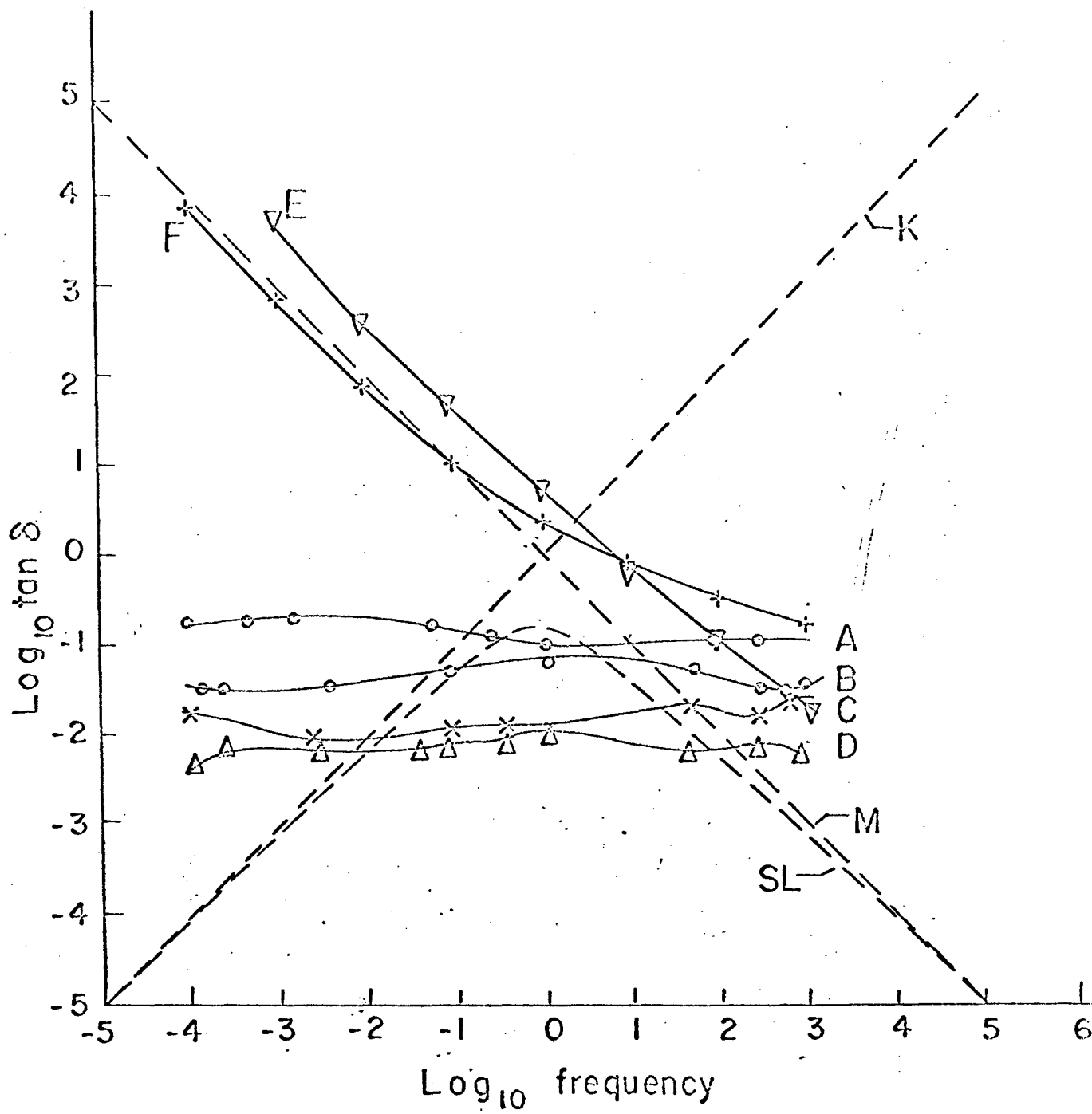
Figure 1

Figure 2

Figure 3

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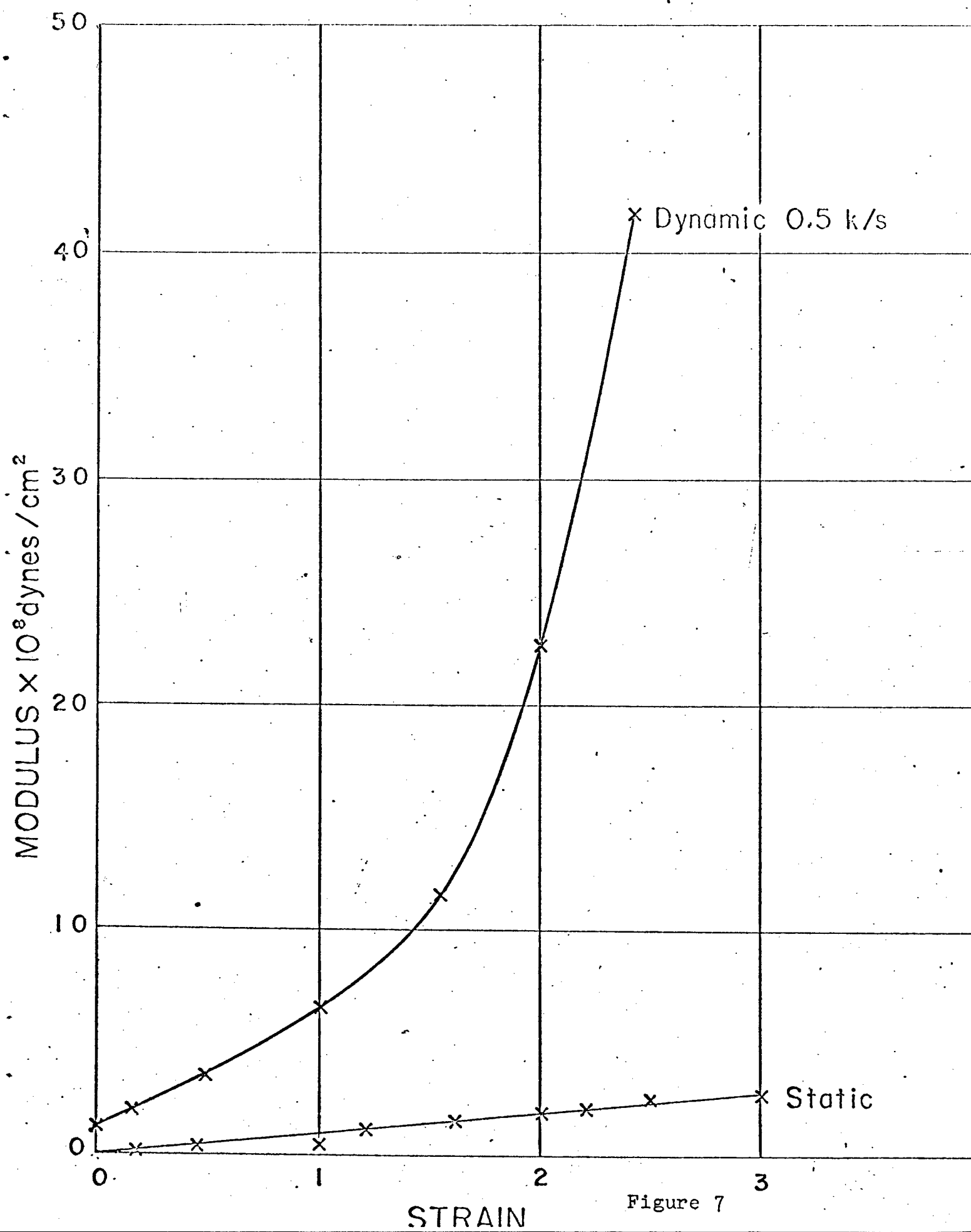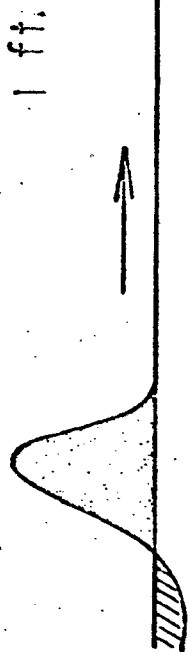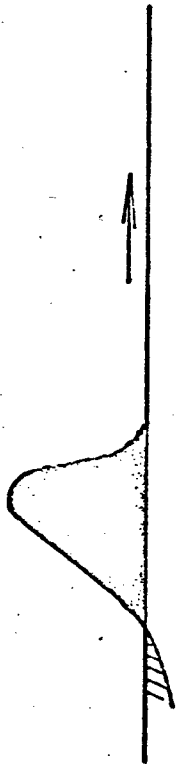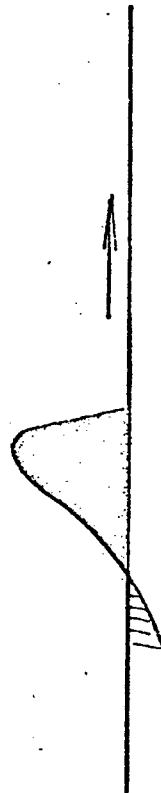
u(t)

μ SEC

CALCULATED

EXPERIMENTAL

Figure 4

38

Figure 5

Figure 6

Figure 7

1 ft.

3 ft.

5 ft.

7 ft.

9 ft.

10

Figure 8

Experiment

Theory

Time from fracture initiation, $\mu$sec. →

Normalized stress or strain